A Short Introduction to Morse Theory

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Contents

These are the notes of my talk about **Morse theory** in the seminar same-named seminar organized by Prof. Dr. S. Sabatini at the university of Cologne. Morse theory is the study of the relations between functions on a space and the shape of the space. In this short introduction we will follow the excellent book of Yukio Matsumoto [1]. Since this is just a short introduction it covers only the first part of the book which deals only with the case of two dimensional spaces, i.e. surfaces.

1 Critical points

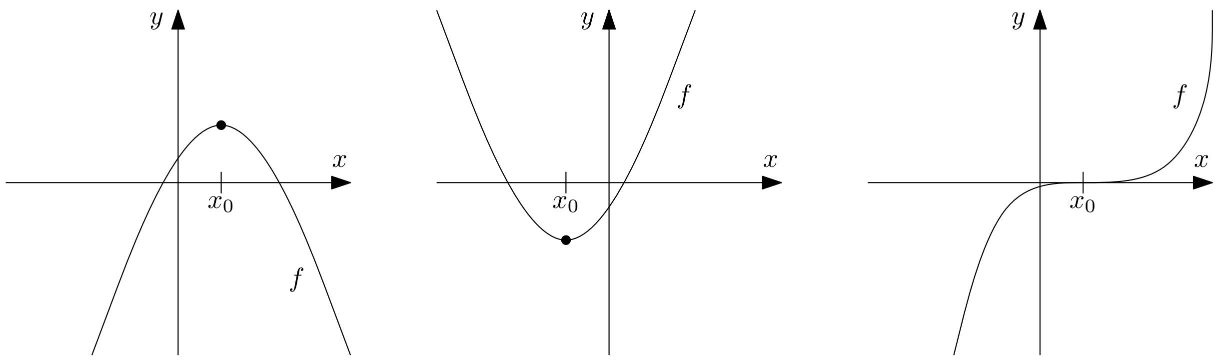
Definitio 1. Let

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) \end{array}$$

be a C^∞ function. A point $x_0 \in \mathbb{R}$ is called a **critical point** of f iff

$$f'(x_0) = \frac{\partial f}{\partial x}(x_0) = 0. \tag{1.1}$$

Notatio 1. A critical pojnts x_0 can be either a (local) maxima, minima or an inflection point of f.



Definitio 2. A critical points x_0 is called **non-degenerate** if $f''(x_0) \neq 0$.

Exemplum 1. Consider the function $f(x) = x^n$ with $n \in \mathbb{N}$. Then we have the following derivatives

$$f'(x) = n \cdot x^{n-1} \quad \text{and} \quad f''(x) = n \cdot (n-1)x^{n-2}. \tag{1.2}$$

Then for $n = 2$ $x_0 = 0$ is a non-degenerate critical point of f. In all other cases, i.e. $n \in \mathbb{N}\setminus\{2\}$, $x_0 = 0$ is a degenerated critical point.

Exemplum 2. Consider the functions $f_1(x) = x^2$, $f_2 = x^3$ and $g = a \cdot x + b$ with $a, b \in \mathbb{R}$. Then f_1 and f_2 have only one and the same critical point $x_0 = 0$. Let us perturb this functions:

$$g_1 := f_1 + g = x^2 + ax + b \quad \text{and} \quad g_2 = f_2 + g = x^3 + ax + b. \tag{1.3}$$

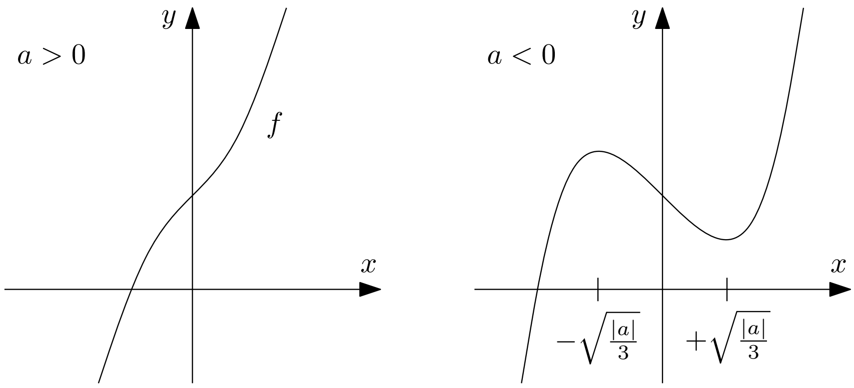
Checking for critical points of g_1:

$$2x + a \stackrel{!}{=} 0 \quad \Leftrightarrow \quad x = -\frac{a}{2}. \tag{1.4}$$

This point is non-degenerated since $g_1''(-\frac{a}{2}) = 2$.
Checking for critical points of g_2:

$$3x^2 + a \stackrel{!}{=} 0 \quad \Leftrightarrow \quad x_0^{\pm} = \pm\sqrt{-\frac{a}{3}}. \tag{1.5}$$

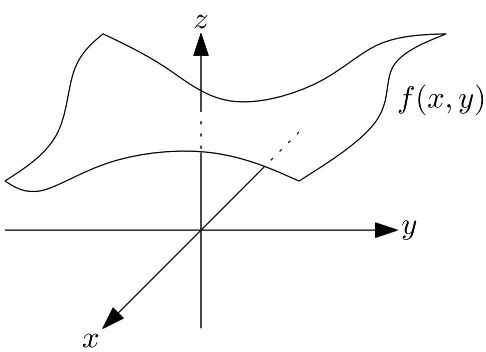
Since $x \in \mathbb{R}$ we have for $a > 0$ no critical point of g_2, but for $a < 0$ we find that $g''(2)(x_0^{\pm}) = \pm 6\sqrt{|a|/3} \neq 0$. Therefore $x_0^{\pm}$ are non-degenerated critical points.



Corollarium 1. *Non-degenerated points are 'stable' under 'perturbations'.*

2 The Hessian

We consider now real-valued function of two variables, i.e. $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$. These function can be visualized as setting $z := f(x, y)$ in a 3D-plot.



Definitio 3. A point $p_0 = (x_0, y_0)$ of a function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is called a **critical points** of f iff

$$\frac{\partial f}{\partial x}(p_0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(p_0) = 0. \tag{2.6}$$

Definitio 4. (i) Let $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ and $p \in \mathbb{R}^2$. The matrix

$$H_f(p) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(p) & \frac{\partial^2 f}{\partial x \partial y}(p) \\ \frac{\partial^2 f}{\partial y \partial x}(p) & \frac{\partial^2 f}{\partial y^2}(p) \end{pmatrix} \tag{2.7}$$

is called the **Hessian** of f at p.

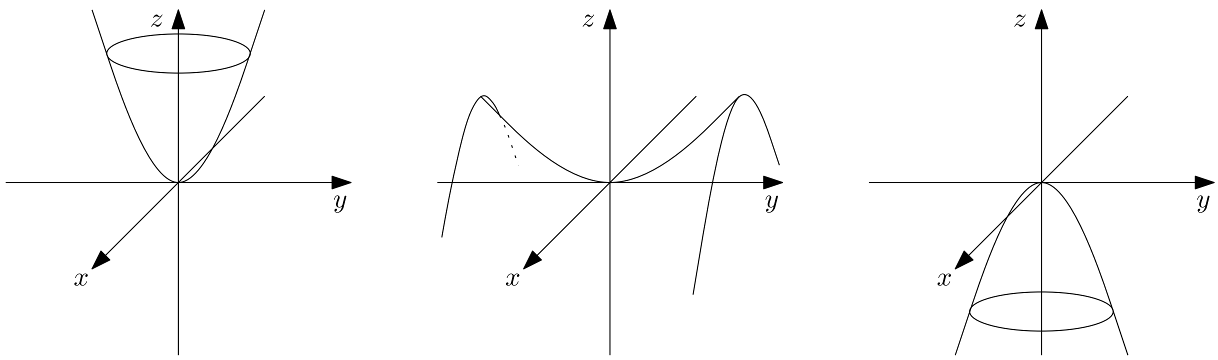
(ii) A critical points p_0 of f is called **non-degenerate** if the determinant of $H_f(p_0)$ is non-zero, i.e.

$$\det H_f(p_0) = \frac{\partial^2 f}{\partial x^2}(p_0) \cdot \frac{\partial^2 f}{\partial y^2}(p_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(p)\right)^2 \neq 0. \tag{2.8}$$

Exemplum 3. Let $f_1(x, y) = x^2 + y^2$, $f_2(x, y) = x^2 - y^2$ and $f_3 = -x^2 - y^2$. Then $p_0 = (0, 0)$ is a critical point of all three functions. Computing the Hessian's:

$$H_{f_1}(p_0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad , \quad H_{f_2}(p_0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad H_{f_3}(p_0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}. \tag{2.9}$$

From this we can see that $\det H_{f_i}(p_0) \neq 0$ for all $i \in \{1, 2, 3\}$ and therefore $p_0 = (0, 0)$ is a non-degenerate critical point of all three functions.



Exemplum 4. Consider the function $f(x, y) = xy$. Then $p_0 = (0, 0)$ is a critical point of f. The Hessian

$$H_f(p_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.10}$$

has non-zero determinant and therefore $p_0 = (0, 0)$ is a non-degenerate critical point of f.

Notatio 2. Consider the function

$$\begin{aligned} \phi : \quad \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \times \mathbb{R} \\ (x, y) &\longmapsto \left(\tfrac{x+y}{2}, \tfrac{x-y}{2}\right). \end{aligned}$$

Then $f(x, y) = (f_2 \circ \phi)(x, y)$. We can see that both functions have the 'same' non-degenerated critical point $p_0 = (0, 0)$. We can make this fact more precise:

Lemma 2. *Let p_0 be a critical point of a function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Consider two sets of coordinates (x, y) and (X, Y) related by a change of coordinates*

$$\begin{array}{rccc} \phi: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (X,Y) & \longmapsto & (x(X,Y), y(X,Y)) \,. \end{array}$$

Denote by $H_f(p_0)$ the Hessian of f computed using coordinates (x, y) and by $H'_f(p_0)$ the Hessian of the same f computed in different coordinates (X, Y). Then the following relation holds:

$$H'_f(p_0) = J_\phi^T(p_0) H_f(p_0) J_\phi(p_0), \tag{2.11}$$

*where $J_\phi(p_0)$ is the so-called **Jacobian-matrix** of ϕ, defined by*

$$J_\phi(p_0) = \begin{pmatrix} \frac{\partial x}{\partial X}(p_0) & \frac{\partial x}{\partial Y}(p_0) \\ \frac{\partial y}{\partial X}(p_0) & \frac{\partial y}{\partial Y}(p_0) \end{pmatrix}. \tag{2.12}$$

Proof. The proof is a simple calculation, where we apply twice the formula for the change of variables in partial derivatives, i.e.

$$\frac{\partial f}{\partial X} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial X} \quad \text{and} \quad \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial Y}. \tag{2.13}$$

Then

$$\frac{\partial^2 f}{\partial X^2} = \frac{\partial}{\partial X}\left(\frac{\partial f}{\partial X}\right) = \frac{\partial}{\partial X}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial X}\right) \tag{2.14}$$

$$= \frac{\partial x}{\partial X}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial X}\right) + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial X^2} + \frac{\partial y}{\partial X}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial X}\right) + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial X^2} = \dots \tag{2.15}$$

This has to be done for all components. Then by evaluating at p_0, i.e. using $\frac{\partial f}{\partial x}(p_0) = \frac{\partial f}{\partial y}(p_0) = 0$, since p_0 is a critical, and comparing the expressions we get the desired result. □

Exemplum 5. Consider again $f_2(x, y) = x^2 - y^2$, $f(x, y) = xy$ and ϕ from remark 2. Then we can compute the associated Jacobian-matrix of ϕ as

$$J\phi(p_0) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = J_\phi^T(p_0). \tag{2.16}$$

Hence we get

$$H'_f(p_0) = J_\phi^T(p_0) H_f(p_0) J_\phi(p_0) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.17}$$

which is the same result as in example 4.

Notatio 3. Since $\det J_\phi(p_0) \neq 0$ for every change of coordinates ϕ we have that from $\det H_f(p_0) \neq 0$ if follows that $\det H_g(p_0) \neq 0$ for all f and g that are related by a coordinate change ϕ.

Corollarium 3. *The property that a critical point p_0 is non-degenerate does not depend on the choice of coordinates. The same is true for degenerate critical points.*

3 The Morse Lemma

Theorema 4. *(The Morse lemma).*
Let p_0 be a non-degenerate critical point of a function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Then we can choose appropriate local coordinates (X, Y) in such a way that the function f expressed with respect to (X, Y) takes one of the following standard forms:

$$\text{(i)} \quad f(X,Y) = X^2 + Y^2 + c, \tag{3.18}$$

$$\text{(ii)} \quad f(X,Y) = X^2 - Y^2 + c, \tag{3.19}$$

$$\text{(iii)} \quad f(X,Y) = -X^2 - Y^2 + c, \tag{3.20}$$

where $c = f(p_0)$ is a constant and $p_0 = (0,0)$ is the origin.

Corollarium 5. *A non-degenerate critical point p_0 of a function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is isolated.*

Proof of Theorem 4. . We choose any local coordinate system (x, y) near the point p_0. Without loss of generality we can assume $p_0 = (0,0)$ in these coordinates. Also we can set $f(p_0) = 0$. We want to show that we can assume

$$\frac{\partial^2 f}{\partial x^2}(p_0) \neq 0. \tag{3.21}$$

If $\frac{\partial^2 f}{\partial x^2}(p_0) \neq 0$ is already true there is nothing left to prove. If on the other hand $\frac{\partial^2 f}{\partial y^2}(p_0) \neq 0$ and $\frac{\partial^2 f}{\partial x^2}(p_0) = 0$ we can interchange the x- and y-axis. So we have to consider only the case where

$$\frac{\partial^2 f}{\partial x^2}(p_0) = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}(p_0) = 0. \tag{3.22}$$

Since p_0 is a non-degenerate critical point of f the Hessian H_f with respect to (x, y) at p_0 is given by

$$H_f(p_0) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \quad \text{with } a \in \mathbb{R} \backslash \{0\}. \tag{3.23}$$

Now we introduce new coordinates (X, Y) via

$$x = X - Y \quad \text{and} \quad y = X + Y. \tag{3.24}$$

The corresponding Jacobian J is given by

$$J = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{3.25}$$

so that the Hessian $H'_f(p_0)$ at p_0 with respect to (X, Y) has the form

$$H'_f(p_0) = J^T H_f(p_0) J = \begin{pmatrix} 2a & 0 \\ 0 & -2a \end{pmatrix} \tag{3.26}$$

by using Lemma 2. Then we have

$$\frac{\partial^2 f}{\partial X^2}(p_0) = 2a \neq 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial Y^2}(p_0) = -2a \neq 0 \tag{3.27}$$

since $a \neq 0$. So we can use the assumption given in equation (3.21).From calculus of several variables we know that for every function $f(x, y)$ near the origin with $f(0,0) = 0$ there are function $g(x, y)$ and $h(x, y)$ such that

$$f(x, y) = xg(x, y) + yh(x, y) \tag{3.28}$$

in some neighborhood of the origin. Then

$$\frac{\partial f}{\partial x}(0,0) = g(0,0) \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = h(0,0). \tag{3.29}$$

Since p_0 is a critical point of f we further have

$$\frac{\partial f}{\partial x}(0,0) = g(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = h(0,0) = 0 \tag{3.30}$$

and this implies that we can apply the above fact again on g and h, i.e. we can write

$$g(x, y) = xh_{11}(x, y) + yh_{12}(x, y) \quad \text{and} \quad h(x, y) = xh_{21}(x, y) + yh_{22}(x, y), \tag{3.31}$$

where $h_{11}, h_{21}, h_{12}, h_{22} \in C^\infty(\mathbb{R}^2, \mathbb{R})$ are suitable functions. Then we can rewrite f as

$$f(x, y) = x^2 h_{11}(x, y) + xy(h_{12} + h_{21}) + y^2 h_{22} = x^2 H_{11}(x, y) = 2xyH_{12} + y^2 H_{22}, \tag{3.32}$$

where in the last step we defined

$$H_{11} := h_{11} \quad , \quad H_{12} := \frac{h_{12} + h_{21}}{2} \quad \text{and} \quad H_{22} := h_{22}. \tag{3.33}$$

Then the Hessian H_f at $p_0 = (0,0)$ of f is given by

$$H_f(p_0) = 2 \begin{pmatrix} H_{11}(0,0) & H_{12}(0,0) \\ H_{12}(0,0) & H_{22}(0,0) \end{pmatrix} \tag{3.34}$$

Due to equation (3.21) we can deduce that $H_{11} \neq 0$ and since H_{11} is continuous we see that $H_{11}(x, y)$ is non-zero in some neighborhood of $(0,0)$. Then we can define a new coordinate

$$X = \sqrt{|H_{11}(x, y)|} \left(x + \frac{H_{12}(x, y)}{H_{11}(x, y)} y \right). \tag{3.35}$$

Leaving y as it is we can compute the Jacobian between (x, y) and (X, y), which is evaluated at the origin non-zero. Therefore (X, y) defines a local coordinate system for some neighborhood of the origin. Squaring X gives us

$$X^2 = |H_{11}| \left(x^2 + 2\frac{H_{12}(x, y)}{H_{11}(x, y)} xy + \left(\frac{H_{12}(x, y)}{H_{11}(x, y)} y \right)^2 \right) \tag{3.36}$$

$$= \begin{cases} H_{11}(x, y)x^2 + 2H_{12}(x, y)xy + \frac{H_{12}^2(x,y)}{H_{11}(x,y)} y^2 & \text{, for } H_{11}(x, y) > 0 \\ -H_{11}(x, y)x^2 - 2H_{12}(x, y)xy - \frac{H_{12}^2(x,y)}{H_{11}(x,y)} y^2 & \text{, for } H_{11}(x, y) < 0 \end{cases} \tag{3.37}$$

Comparing this result with (3.32) we get

$$f(X, y) = \begin{cases} X^2 + Ky^2 & \text{, for } H_{11} > 0 \\ -X^2 + Ky^2 & \text{, for } H_{11} < 0 \end{cases}, \tag{3.38}$$

where we have set $K := \left(H_{22} - \frac{H_{12}^2}{H_{11}}\right)$. Choosing as a new coordinate

$$Y = \sqrt{|K|}\, y. \tag{3.39}$$

we can rewrite (3.382), such that f has the following expressions in the new local coordinates (X, Y):

$$f(X,Y) = \begin{cases} X^2 + Y^2 & \text{, for } H_11 > 0,\ K > 0 \\ X^2 - Y^2 & \text{, for } H_11 > 0,\ K < 0 \\ -X^2 + Y^2 & \text{, for } H_11 < 0,\ K > 0 \\ -X^2 - Y^2 & \text{, for } H_11 < 0,\ K < 0 \end{cases} \tag{3.40}$$

□

Notatio 4. By interchanging X and Y we can see that the cases $f(X,Y) = X^2 - Y^2$ and $f(X,Y) = -X^2 + Y^2$ are essentially the same standard form.

Definitio 5. Let p_0 be a non-degenerate critical point of a function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$. By using theorem 4 we can choose suitable coordinates (x, y) in some neighborhood of p_0 such that f has one of the above standard forms. A **index** of a non-degenerated critical point p_0 of f is 0, 1 or 2 if f has the standard form $f = x^2 + y^2 + c$, $f = x^2 - y^2 + c$ or $f = -x^2 - y^2 + c$ respectively, i.e. the index of p_0 is the number of minus signs in the f.

Notatio 5. We can think of standard forms of a function f as the ones, where the Hessian has one of the following forms:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad , \quad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}. \tag{3.41}$$

Then Sylvester's law in linear algebra tells us that the number of minus signs in $H_f(p_0)$ does not depend on the chosen coordinate change. This proves that the index of a non-degenerate critical point is well-defined.

References

[1] Yukio Matsumoto. *An Introduction to Morse Theory.* Oxford University Press, 2001.